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An Estimate of Stream Speeds About an Elliptic Cylinder

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AN ESTIMATE OF STREAM SPEEDS ABOUT AN ELLIPTIC CYLINDER

I Summary

The main results are shown in the graphs. The first three of these relate to a particular case of elliptic dimensions and stream velocity; their value is essentially in Graph 3 which shows the variation of speed with position about the particular ellipse. The pattern of speeds is shown by means of constant speed lines, which are analagous to lines of constant elevation on a contour map. A knowledge of a typical speed pattern and the extreme speeds attained for cases of different ellipse dimensions enables one to estimate stream speeds at different points for these cases. Graph 4 gives the maximum speed as a function of stream speed and ellipse dimensions.

II Theory

1. Let there be a simple closed curve C, representing a body, about which there is two-dimensional motion of an inviscid liquid in the complex z-plane. This motion is specified by the potential function

$$\omega = \varphi + i\psi$$

where $\varphi(x, y)$ is the velocity potential and $\psi(x, y)$ the stream function. At a point on C the velocity component normal to C is zero. The φ-constant curves are orthogonal to C as well as the ψ-constant curves. Suppose f is a conformal mapping of the z-plane onto the ζ-plane. Then the curves $f(C)$ and $f(\psi\text{-constant})$

are orthogonal to the curves $f(\varphi\text{-constant})$, and $f(\varphi + i\psi)$ is the potential function for motion about the curve f(C) in the new plane.

2. Our procedure for finding the potential function for flow about a given curve is to find a conformal mapping from a known flow and curve to the desired flow and curve. The unknown potential function is simply the transformed known potential function.

3. A particularly simple known flow is that of a uniform undisturbed stream. Appropriately oriented in the z-plane, its potential function is

$$w = \varphi + i\psi = U(x + iy)$$

where U is the constant velocity of flow parallel to the real axis

4. For motion about an ellipse, C is taken as

$$\frac{x^2}{c^2 \cosh^2 \xi_0} + \frac{y^2}{c^2 \sinh \xi_0} = 1$$

The semi-major axis is $a = c \cosh \xi_0$; the semi-minor axis is $b = c \sinh \xi_0$; and $2c$ is the distance between foci. ξ_0 is determined from a and b by

$$\xi_0 = \frac{1}{2} \log \frac{a+b}{a-b} .$$

5. Consider a transfo tion from the $z = x + iy$ plane to the $\zeta = \xi + i\eta$ plane followed by a transformation from the ζ-plane to the $w = \varphi + i\psi$ plane. Let these transformations be given by

$$z = x + iy = c \cosh \zeta \qquad (1)$$

$$w = \varphi + i\psi = U(a+b) \cosh(\zeta - \xi_0) \qquad (2)$$

In the z-plane the ellipse with semi-axes a and b determines the constant ξ_0 and maps into the straight line $\xi_0 + i\eta$ in the ζ -plane under the transformation (1). Now $\zeta = \xi_0 + i\eta$ in (2) is mapped into the real axis $i\psi = 0$ of the w -plane. Hence motion in the w-plane is undisturbed; stream lines in this plane are ψ = constant and equi-potential lines are φ = constant. The mappings of these lines in the z-plane give the stream lines and equi-potential lines for the motion about the body represented by the ellipse ξ_0.

6. Let u and v be the x and y components of velocity in the z-plane. Then

$$\frac{dw}{dz} = \frac{dw}{dz}\frac{dz}{dx} = \frac{\partial w}{\partial x} = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = -u + iv.$$

Thus the square of speed is the product of $\frac{dw}{dz}$ by its conjugate:

$$q^2 = (u - iv)(u + iv) = \frac{dw}{dz}\cdot\frac{d\bar{w}}{d\bar{z}}.$$

For our case

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz}$$

and

$$\frac{dw}{dz} = \frac{U(a+b)\sinh(\zeta - \xi_0)}{c\sinh\zeta}$$

Speed is then

$$q^2 = \frac{U^2(a+b)}{a-b}\cdot\frac{\sinh^2(\xi-\xi_0) + \sin^2\eta}{\sinh^2\xi + \sin^2\eta}.$$

7. Differentiation with respect to η gives the maximum at $\eta = \frac{\pi}{2}$.

8. The maximum stream speeds of Graph 4 occur at the tips of the minor axis.

III Illustration

1. Suppose it is desired to find the stream speed at a point on $\eta = 90°$, and a distance of 50 ft from the center of the ellipse where the undisturbed stream speed $U = 10$ ft/sec and the ellipse major axis is 450 ft, and the minor axis 25 ft. Then $a = 450/2$ and $b = 25/2$ and

$$\frac{a+b}{a-b} = 1.118$$

Graph 4 shows the ratio of maximum speed to undisturbed stream speed for these dimensions to be 1.033. This gives

$$q = 1.033(10) = 10.33 \text{ ft/sec}$$

at the end of the minor axis. In the region considered we see on Graph 3 that stream speed drops about 1/4 of the difference between the maximum stream speed and the undisturbed stream speed. At the given point then we obtain the estimate

$$10.33 - 1/4(0.33) = 10.09 \text{ ft/sec.}$$

2. Although our model is two-dimensional, these results are in qualitative agreement with results of the analyses of speeds about a ship contained in the report, "The Flow Field Surrounding a Moving Vessel," by Margenau, Hellerman, and Van Zandt, (HPP:TR 13). In that report, point and line sources and sinks were used to model the ship, and the flow calculations were made in three dimensions.

IV Reference

Milne-Thomson, _Theoretical Hydrodynamics_, Macmillan and Co., 1938, Chapter VI.

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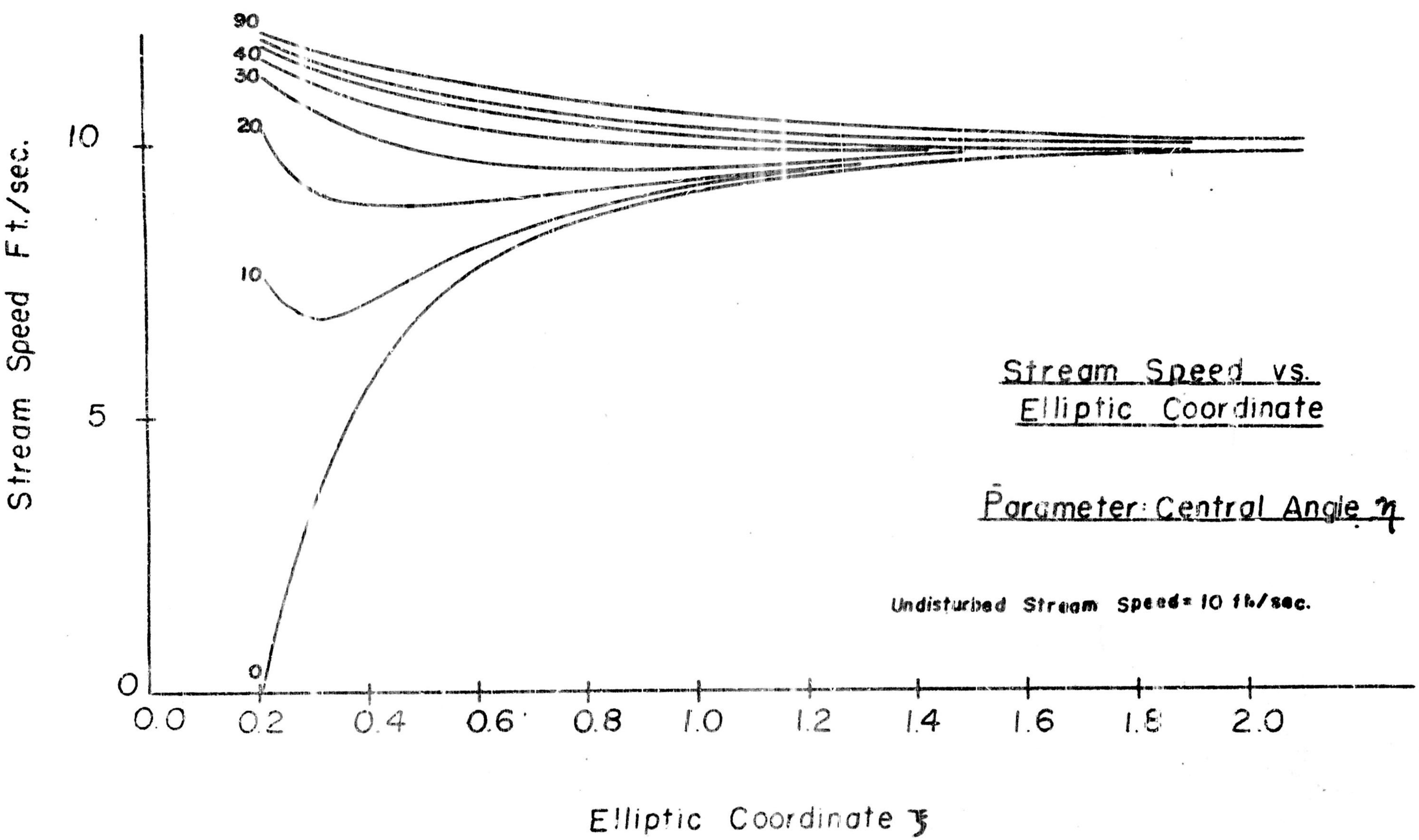
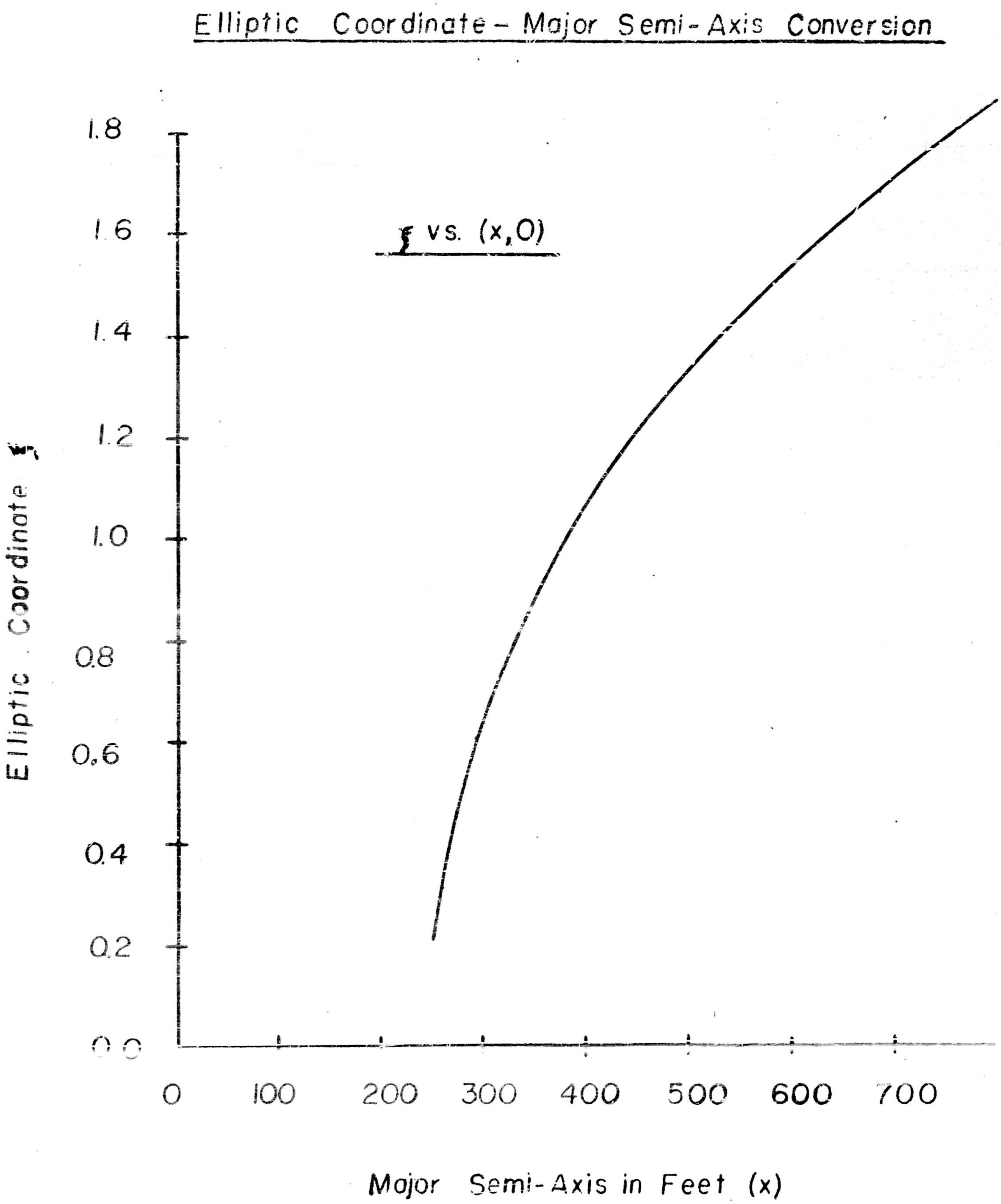
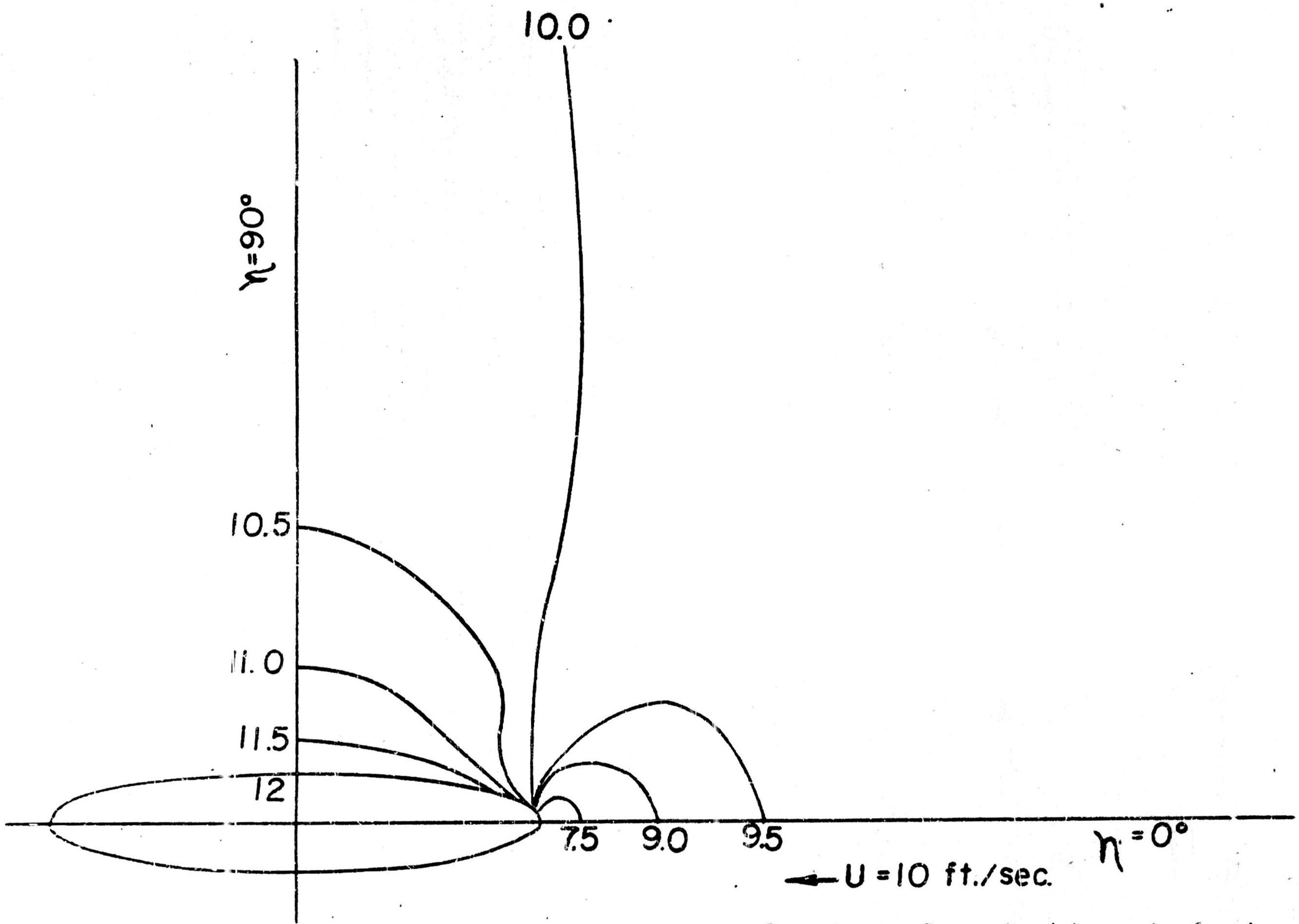

Stream Speed Ft./sec.
10
5
0
90
40
30
20
10
0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
Stream Speed vs.
Elliptic Coordinate
Parameter: Central Angle η
Undisturbed Stream Speed = 10 ft/sec.
Elliptic Coordinate ξ

Graph 1



Elliptic Coordinate - Major Semi-Axis Conversion
ξ vs. (x,0)
Elliptic Coordinate ξ
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
700
Major Semi-Axis in Feet (x)

Graph 2



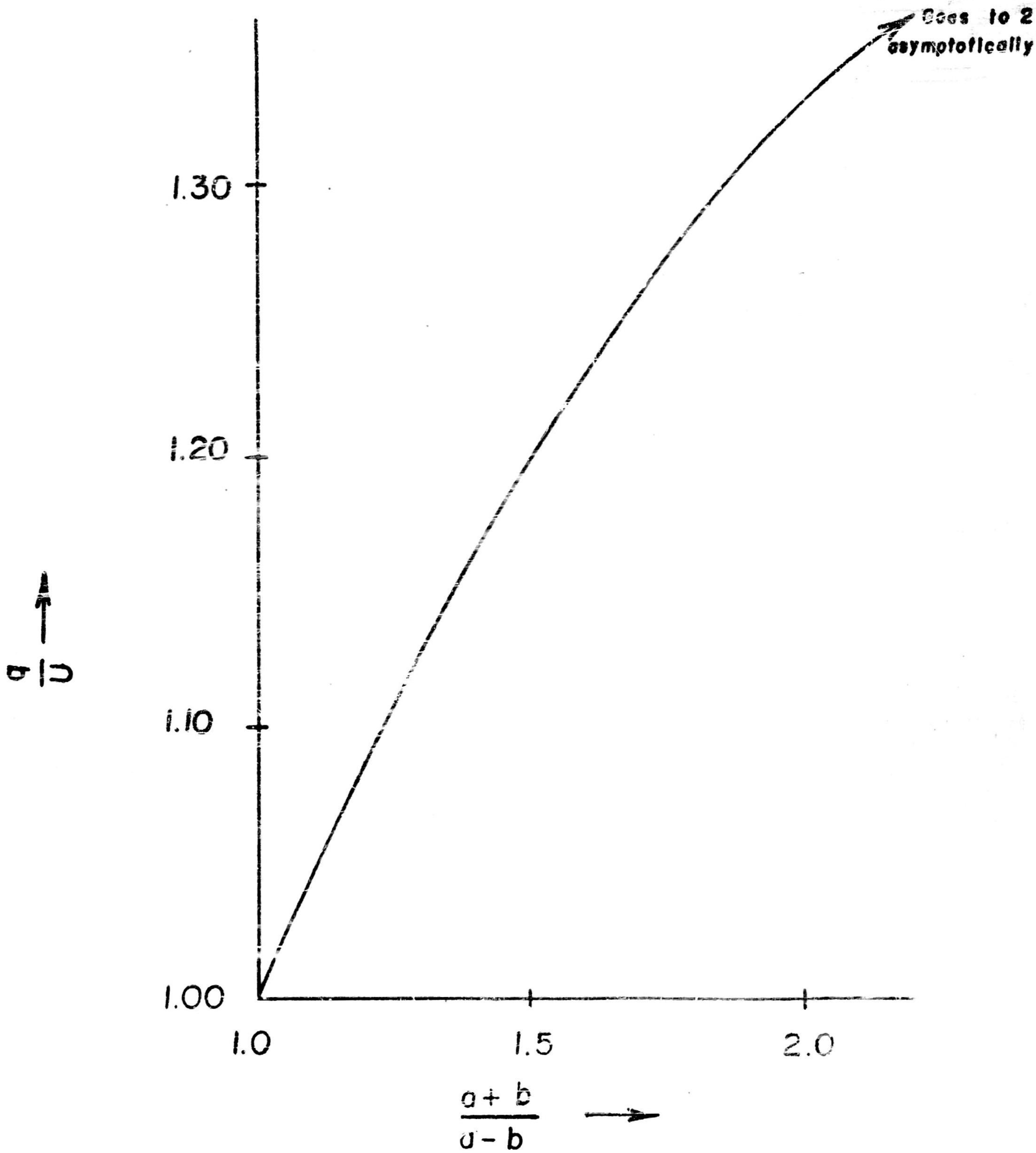
Constant Speed Lines in feet per sec.

stream velocity: 10 ft./sec.
Elliptic cylinder major axis: 500 ft.
minor axis: 100 ft.

Scale: 0 50' 100' 150' 200'

Graph 3

$\dfrac{\text{Maximum Speed}}{\text{Stream Speed}}$ vs. Ellipse Dimensions



a semi-major axis
b semi-minor axis
q maximum speed
U stream speed

Graph 4